Distributivity and modularity in varieties of algebras

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1. Introduction. A variety $\mathfrak{V}$ of algebras is called *distributive*, if for every algebra $A \in \mathfrak{V}$ the lattice $\Theta(A)$ of all congruence relations over A is distributive. Distributivity of a variety will be denoted by $\Delta(\mathfrak{V})$. B. JÓNSSON [1] has shown the following theorem:

Let $\mathfrak{V}$ be a variety of algebras. $\Delta(\mathfrak{V})$ is valid if and only if for some integer $n \geqq 2$ the following condition holds. $\Delta_n(\mathfrak{V})$: There exist ternary (derived) operations $\tau_0, \tau_1, \ldots, \tau_n$, such that for $i=0, 1, \ldots, n-1$ the identities

$$\tau_0(x, y, z) = x, \quad \tau_n(x, y, z) = z, \quad \tau_i(x, y, x) = x$$

$$\tau_i(x, x, z) = \tau_{i+1}(x, x, z) \quad (i \text{ even}), \qquad \tau_i(x, z, z) = \tau_{i+1}(x, z, z) \quad (i \text{ odd})$$

hold in every member of $\mathfrak{V}$.

If $\Delta_n(\mathfrak{V})$ is valid, we say that the variety $\mathfrak{V}$ is *n-distributive*. Evidently, $\Delta_n(\mathfrak{V})$ implies $\Delta_{n+1}(\mathfrak{V})$, because we can define $\tau_{n+1} = \tau_n$. B. JÓNSSON has shown in [1] that $\Delta_3(\mathfrak{V})$ does not imply $\Delta_2(\mathfrak{V})$, and G. GRÄTZER asks in [2] for examples which show that $\Delta_n(\mathfrak{V})$ does not imply $\Delta_{n-1}(\mathfrak{V})$ for $n \geqq 3$. We prove this suggestion by methods previously applied by the author in [3].

We can answer also the analogous question for A. DAY's characterization of modularity [4].

Our terminology and notation are essentially those of [5].

2. Distributivity. We have the following:

Theorem. *For each integer $n \geqq 2$ there exists a variety which is $(n+1)$-distributive, but not n-distributive.*

Let the variety $\mathfrak{V}$ be defined by ternary basic operations $\tau_1, \tau_2, \ldots, \tau_n$ and the following identities

(1) $$\tau_i(x, y, x) = x \quad (i = 1, 2, \ldots, n),$$

(2) $$x = \tau_1(x, x, z),$$

$$(3)\qquad \begin{cases} \tau_i(x, z, z) = \tau_{i+1}(x, z, z) & (i=1, 3, 5, \ldots), \\ \tau_i(x, x, z) = \tau_{i+1}(x, x, z) & (i=2, 4, 6, \ldots), \end{cases}$$

$$(4)\qquad \tau_n(x, x, z) = z \,(\text{if } 2/n) \quad \text{or} \quad \tau_n(x, z, z) = z \,(\text{if } 2\nmid n).$$

We prove that $\mathfrak{V}$ is $(n+1)$-distributive, but not n-distributive.

By *word* we will mean always a $\langle \tau_1, \ldots, \tau_n \rangle$-word in the alphabet $\langle x, y, z \rangle$. Two words are called *equal* if they coincide as rows. A word u is a *subword* of v if u is an interval in v; if u is not equal to v, then it is a *proper* subword. A word w of the form $w = \tau_i(u, \bar{u}, \bar{\bar{u}})$, $(1 \leq i \leq n)$ is called a τ_i-*word.* We say that the word w_1 is a *reduction* of the word $w_2 = \tau_i(u_2, \bar{u}_2, \bar{\bar{u}}_2)$, if $w_1 = w_2$ is an identity in $\mathfrak{V}$ (i.e., $w_1 = w_2$ holds in $\mathfrak{V}$) and w_1 is equal to at least one of the words u_2, $\bar{\bar{u}}_2$. A word $w = \tau_i(u, \bar{u}, \bar{\bar{u}})$ is called *reduced,* if neither $w = u$ nor $w = \bar{\bar{u}}$ are identities in $\mathfrak{V}$.

Let (w_1, w_2) be a pair of words such that w_1 changes into w_2 by a single application of any of the identities from (1)—(4) to the whole word w_1 or to a subword of w_1. Then we can distinguish between three types of such pairs:

Type 1: w_1 is a reduction of w_2.

Type 2: w_2 is a reduction of w_1.

Type 3: all other cases.

Proposition. *The pair of words is of Type* 3 *means:*

(i) w_1 *and* w_2 *are either both reduced or none of them is reduced,*

(ii) w_1 *moves into* w_2, *if we apply one of the identities* (3) *to the whole word* w_1 *or any identity from* (1)—(4) *to one of the proper subwords of* w_1.

The sequence of words $w_1, w_2, \ldots, w_m$ is called *simple,* if every term of the sequence is either equal to the next one or changes into it by a single application of one of the identities (1)—(4). A simple sequence $w_1, \ldots, w_m$ of words is called *minimal simple,* if every simple sequence which begins with w_1 and ends with w_m has at least m terms.

Lemma 1. *Let* $w_1, \ldots, w_s$ $(s>2)$ *be such a simple sequence of words that* w_1 *is a reduction of* w_2, *the word* w_s *a reduction of* w_{s-1} *and every pair* (w_{j-1}, w_j) $(j=3, \ldots, s-1)$ *is of Type 3. Then this sequence is not minimal simple.*

Proof. The word w_1 changes into w_2 by one of the identities (1), (2) or (4). By one of the same identities w_{s-1} changes into w_s. For $j=2, 3, \ldots, s-1$ the w_j are τ_{i_j}-words: $w_j = \tau_{i_j}(u_j, \bar{u}_j, \bar{\bar{u}}_j)$. In all the possible cases we can find a simple sequence beginning with w_1, ending with w_s and consisting of $s-2$ terms. The table below gives such a sequence in the following way: Let for instance w_1 move into w_2 by (1) and w_{s-1} into w_s by (2). Then w_1 is equal to u_2, and u_{s-1} is equal to w_s. The sequence $S_1: w_1=u_2, u_3, \ldots, u_{s-2}, u_{s-1}=w_2$ evidently consists of $s-2$ terms, and by (ii) in the proposition above it is simple because the (w_{j-1}, w_j) are of Type 3.

$w_{s-1} \to w_s$ by / $w_1 \to w_2$ by	(1)	(2)	(4)
(1)	S_1	S_1	S_2
(2)	S_1	S_1	S_3
(4)	S_2	S_4	S_2

S_1: $w_1, u_3, u_4, \ldots, u_{s-2}, w_s$.

S_2: $w_1, \bar{\bar{u}}_3, \bar{\bar{u}}_4, \ldots, \bar{\bar{u}}_{s-2}, w_s$.

S_3: $w_1, \bar{u}_3, \bar{u}_4, \ldots, \bar{u}_{s-2}, w_s$ (n odd).

$w_1, \bar{u}_3, \ldots, \bar{u}_k, \bar{\bar{u}}_{k+1}, \ldots, \bar{\bar{u}}_{s-2}, w_s$ (n even).

Here k is an integer such that $1 \leqq k < s$, $i_k = n-1$, $i_{k+1}=n$. This means $w_k = \tau_{n-1}(u_k, \bar{u}_k, \bar{\bar{u}}_k)$, $w_{k+1}=\tau_n(u_{k+1}, \bar{u}_{k+1}, \bar{\bar{u}}_{k+1})$ and $\bar{u}_k, \bar{\bar{u}}_k, \bar{u}_{k+1}, \bar{\bar{u}}_{k+1}$ are equal. S_4 is the reverse of S_3, where S_3 is formed from $w_s, \ldots, w_1$ instead of $w_1, \ldots, w_s$.

Lemma 2. *If $w=\tau_i(u, \bar{u}, \bar{\bar{u}})$ and $w'=\tau_{i'}(u', \bar{u}', \bar{\bar{u}}')$ are reduced words such that $w=w'$ is an identity in $\mathfrak{V}$, then* (i) *the difference between indices i and i' is at most* 1, (ii) *the identities $u=u'$, $\bar{u}=\bar{u}'$, $\bar{\bar{u}}=\bar{\bar{u}}'$ hold in $\mathfrak{V}$, too.*

Proof. Let $w=w_1, w_2, \ldots, w_m=w'$ be a minimal simple sequence. Such a sequence exists for any words w, w' provided $w=w'$ is an identity in $\mathfrak{V}$.

Suppose all pairs (w_j, w_{j+1}), $(j=1, 2, \ldots, m-1)$ are of Type 3. Then it follows from (ii) in the proposition above that for every j the equations

$$(5) \qquad u_j = u_{j+1}, \quad \bar{u}_j = \bar{u}_{j+1}, \quad \bar{\bar{u}}_j = \bar{\bar{u}}_{j+1} \qquad (j = 1, 2, \ldots, m-1)$$

are identities in $\mathfrak{V}$. Assertion (ii) of the lemma follows obviously. To prove (i) we show that $|i-i'| > 1$ implies that $u=\bar{\bar{u}}$ is an identity in $\mathfrak{V}$. Then with respect to (1), w cannot be reduced in contradiction to the supposition.

Let for any j the pair $(w_j, w_{j+1}) = (\tau_{i_j}(u_j, \bar{u}_j, \bar{\bar{u}}_j), \tau_{i_{j+1}}(u_{j+1}, \bar{u}_{j+1}, \bar{\bar{u}}_{j+1}))$ be of Type 3 and the indices i_j and i_{j+1} be different. By definition of Type 3 this difference is 1. If the smaller one of these two integers i_j, i_{j+1} is odd, it follows by (3) that $\bar{u}_j=\bar{\bar{u}}_j$, which by (5) implies that $\bar{u}=\bar{\bar{u}}$ are identities in $\mathfrak{V}$. If the smaller one of i_j, i_{j+1} is even, we can see in the same way that $u=\bar{u}$ is an identity in $\mathfrak{V}$. If now $|i-i'| > 1$, then there are pairs (w_j, w_{j+1}) of the first and of the second kind as well. Hence $u=\bar{u}=\bar{\bar{u}}$ holds in $\mathfrak{V}$.

To complete the proof, it is enough to show that all pairs (w_j, w_{j+1}) $(1 \leqq j < m)$ are of Type 3. If there exists a pair (w_j, w_{j+1}) of Type 2, then, by definition, w_j is not reduced. But w is reduced; hence there are pairs of Type 1, too. Let r be the largest number such that (w_r, w_{r+1}) is of Type 1 and s the smallest number for which (w_{s-1}, w_s) is of Type 2 and $r < s \leqq m$. By Lemma 1 the sequence $w_r, \ldots, w_s$ is not minimal simple in contradiction to the minimal simplicity of $w_1, \dots, w_s$. Similarly we arrive at a contradiction if we suppose the existence of a pair of Type 1 in our sequence.

For any sequence of words $w_1, \dots, w_m$ we will use the following notations:

(6) $\quad w_j^* = w_j(x, x, z) \quad (j=1, 3, 5, \dots), \qquad w_j^* = w_j(x, z, z) \quad (j=2, 4, 6, \dots),$

(7) $\quad w_j^{**} = w_j(x, z, z) \quad (j=1, 3, 5, \dots), \qquad w_j^{**} = w_j(x, x, z) \quad (j=2, 4, 6, \dots),$

(8) $\quad w_j^{***} = w_j(x, y, x) \quad (j=1, 2, \dots, m).$

Lemma 3. *Let w be a τ_i-word: $w=\tau_i(u, \bar{u}, \bar{\bar{u}})$, $(1 \leqq i \leqq n)$. If one of the following identities on the left side hold in $\mathfrak{V}$, the corresponding identities on the right side hold in $\mathfrak{V}$, too:*

$$\text{(i)}\quad w^* = x \Rightarrow \begin{cases} u^* = x & (i = 1, 2, \dots, n-1), \\ \bar{\bar{u}}^* = x & (i = 2, 3, \dots, n). \end{cases}$$

$$\text{(ii)}\quad w^{**} = z \Rightarrow \begin{cases} u^{**} = z & (i = 1, 2, \dots, n-1), \\ \bar{\bar{u}}^{**} = z & (i = 2, 3, \dots, n). \end{cases}$$

$$\text{(iii)}\quad w^{***} = x \Rightarrow \begin{cases} u^{***} = x & (i = 1, 2, \dots, n-1), \\ \bar{\bar{u}}^{***} = x & (i = 2, 3, \dots, n). \end{cases}$$

Proof. (i) If $w^*=x$ is an identity in $\mathfrak{V}$, there is a minimal simple sequence $w^*=w_1, w_2, \dots, w_r=x$. The form of w_1 and w_r involves that in this sequence there exists a term w_s such that w_{s+1} is a reduction of w_j: let s be the smallest index with this property. With respect to Lemma 1, all pairs (w_j, w_{j+1}) for $1 \leqq j < s$ must be of Type 3 and therefore the identities $u_1=u_2=\dots=u_s$, $\bar{\bar{u}}_1=\bar{\bar{u}}_2=\dots=\bar{\bar{u}}_s$ hold in $\mathfrak{V}$. By definition of s, the word w_{s+1} is a reduction of w_s. If $1 \leqq i < n$, it follows by (1), (2) that u_s and w_{s+1} are equal. If $1 < i \leqq n$, it follows by (1), (4) that $\bar{\bar{u}}_s$ and w_{s+1} are equal. In the first case $u^*=u_1=u_2=\dots=u_s=w_{s+1}=x$ and in the second case $\bar{\bar{u}}^*= =\bar{\bar{u}}_0=\dots=\bar{\bar{u}}_s=w_{s+1}=x$ hold in $\mathfrak{V}$.

If $i_s=i$, assertion (i) is shown. If $i_s \neq i$, there is at least one $j < s$ such that one of the identities (3) applied to the whole word w_j yields w_{j+1}. Together with the move of w_s to w_{s+1}, this fact implies that for every $j \leqq s$ the equation $u_j=\bar{u}_j=\bar{\bar{u}}_j$ is an identity and therefore $u^*=x=\bar{\bar{u}}^*$ is an identity in $\mathfrak{V}$, too. Assertion (i) is proved.

To prove (ii) we replace in the proof of (i) x by z and all signs with one star by the same sign with two stars. Correspondingly, to prove (iii) we replace in the proof of (i) all signs with one star by the same with three stars.

Proof of theorem. It is clear by (1)—(4) that $\mathfrak{V}$ is $(n+1)$-distributive. We will prove that if $\mathfrak{V}$ is $(m+1)$-distributive then $m \geqq n$. Using the notations (6)—(8) given above, $(m+1)$-distributivity means that there is a sequence of words $v_1, v_2, \dots, v_m$ such that

(9) $$v_j^{***} = x \qquad (j=1, 2, \ldots, m),$$

(10) $$x = v_1^*,$$

(11) $$v_j^{**} = v_{j+1}^* \qquad (j = 1, m-1),$$

(12) $$v_m^{**} = z$$

are identities in $\mathfrak{V}$.

Suppose that for a fixed integer $m \geqq 1$ among all such sequences the sequence $w_1, w_2, \ldots, w_m$ has the smallest total number of operator symbols τ_i $(1 \leqq i \leqq n)$. Let r be the largest index such that $1 \leqq r \leqq m$ and w_r^* has a reduction. Further, let s be the smallest index such that $r \leqq s \leqq m$ and w_s^{**} has a reduction. It is clear that such indices r, s exist, because w_1^* and w_m^{**} have reductions.

We distinguish 3 cases:

(α) there is no τ_n-word among $w_r, \ldots, w_s$,

(β) there is no τ_1-word among $w_r, \ldots, w_s$,

(γ) among $w_r, \ldots, w_s$ there are τ_1-words and τ_n-words as well.

In case (α) and (β) we take the sequences

$$w_1, \ldots, w_{r-1}, \quad u_r, \ldots, u_s, \quad w_{s+1}, \ldots, w_m$$

and

$$w_1, \ldots, w_{r-1}, \quad \bar{\bar{u}}_r, \ldots, \bar{\bar{u}}_s, \quad w_{s+1}, \ldots, w_m, \text{ respectively.}$$

In both cases the new sequence evidently consists of m words and has less operator symbols τ_i than the sequence $w_1, \ldots, w_m$. Moreover, the identities (9)—(12) hold for the new sequence, too. Indeed, the identity (9) follows from (iii) in Lemma 3. If $r=1$, then (10) follows from (i) in Lemma 3, and if $r>1$, then (10) is the same as in the given sequence. The identities (11) follows from the definition of the reduction and from Lemma 2. Finally, (12) follows either from (ii) of Lemma 3) (if $s=m$) or it is the same as in the sequence $w_1, \ldots, w_m$ (if $s<m$). Thus we got a contradiction to the minimum property of $w_1, \ldots, w_m$.

In case (γ) the words $w_{r+1}^*, w_{r+2}^*, \ldots, w_s^*$; $w_r^{**}, w_{r+1}^{**}, \ldots, w_{s-1}^{**}$ are reduced by supposition, and (11) states that the identities $w_j^{**} = w_{j+1}^*$ $(j=r, r+1, \ldots, s-1)$ hold in $\mathfrak{V}$. By (i) in Lemma 2 it follows that if w_j is an τ_{i_j}-word and w_{j+1} is an $\tau_{i_{j+1}}$-word, then $|i_j - i_{j+1}| \leqq 1$. But among the words $w_r, \ldots, w_s$ there are τ_1-words and τ_n-words as well. Hence there are τ_i-words for $i=1, 2, \ldots, n$. It follows that $s-r+1 \geqq n$ and therefore $m \geqq n$, q.e.d.

3. Modularity. Let $\Sigma(\mathfrak{V})$ denote the property of the variety $\mathfrak{V}$ that for every algebra $A \in \mathfrak{V}$ the lattice $\theta(A)$ of all congruence relations over A is modular. A. DAY [4] has shown the following theorem:

Let $\mathfrak{V}$ be a variety of algebras. $\Sigma(\mathfrak{V})$ is valid if and only if for some integer $n \geqq 2$ the following holds. $\Sigma_n(\mathfrak{V})$: There exist 4-ary operations $\mu_0, \mu_1, \ldots, \mu_n$ such that for $i=0, 1, \ldots, n-1$ the identities

$$\mu_0(x, y, z, w) = z, \quad \mu_n(x, y, z, w) = w, \quad \mu_i(x, y, y, x) = x,$$

$$\mu_i(x, y, y, w) = \mu_{i+1}(x, y, y, w) \qquad (i \text{ odd}),$$

$$\mu_i(x, x, w, w) = \mu_{i+1}(x, x, w, w) \qquad (i \text{ even})$$

hold in every member of $\mathfrak{V}$.

If $\Sigma_n(\mathfrak{V})$ is valid, the variety $\mathfrak{V}$ is called n-modular. It is evident that $\Sigma_n(\mathfrak{V})$ implies $\Sigma_{n+1}(\mathfrak{V})$. The fact that $\Sigma_{n+1}(\mathfrak{V})$ does not imply $\Sigma_n(\mathfrak{V})$ can be proved in a way analogous to that of the proof concerning distributivity. In other words we have the following

Theorem. *For each integer $n \geqq 2$ there exists a variety which is $(n+1)$-modular, but not n-modular.*

We omit the proof, because it coincides essentially with the proof of the preceding theorem. Of course, there are some differences which are due to the parity of the operations μ_i, but they are only of formal nature.

Problems. Find finite algebras A_m $(m=2, 3, \ldots)$ such that any variety $\mathfrak{V}$ which contains A_m can be $(m+1)$-distributive but not m-distributive. Solve the same problem for modularity. Probably, by such examples the proofs in this paper may be shortened. In the case of varieties with ideals and varieties which are $(m+1)$-permutable but not m-permutable such finite examples are given by A. F. Mutylin [6] and E. T. Schmidt [7], respectively.

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